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Some properties of Fundamental Complex functions

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Abstract:
The purpose of this work is to define and study a new type of functions which have only essential singular points on a region in complex plane, so that several properties of this type of functions are given. Also we study the set of this type of functions algebraically to get some theorems and results.

Keywords:
Fundamental complex functions, essential singular points , removable singular points , poles , analytic functions.

Introduction:

The isolated singular points of a complex valued function was classified in [1] into three kinds: removable singular points , poles and essential singular points.

A function that has only poles on a region in the complex plane C is called meromorphic function which was studied in [2-5].

An Elliptic function was studied in [6] which is a meromorphic doubly periodic function.

Here we study the functions that have only essential singular points on a region in the complex plane. Before studying this type of functions we have to mention some elementary definitions and notations as follows:

Definitions:

1. An isolated singular point z_o of a function f is said to be an essential singular point if the principal part of Laurent series of f about z_o contains an infinite number of terms.

2. A neighborhood $N(z_o, r)$ is the set of all complex numbers z such that $|z - z_o| < r$.

3. A point z_o is said to be an interior point of a set D if there exists a neighborhood $N(z_o, r)$ such that $N(z_o, r) \subseteq D$.

4. A point z_o is said to be boundary point of a set D if every neighborhoods $N(z_o, r)$ of z_o contains a point of D and a point of D^c.

5. The coefficient of $(z - z_o)^{-1}$ in Laurent series of $f(z)$ about z_o is called residue of f at z_o , which is written as $\operatorname{Re} s(f, z_o)$.

Notations:

1. We denote the set of analytic functions , Meromorphic functions and harmonic functions on a region D by $A(D)$, $M(D)$ and $H(D)$ respectively.

2. We denote the set of zeros , poles and essential singular points of f on a region

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D by $N(f)$, $P(f)$ and $E(f)$ respectively.

3. We denote the set of interior points, boundary points and limit points by $Int(D)$, ∂D and $d(D)$ respectively.

In this work we study the set of functions that have only essential singular points on a region $D \subseteq C$.

Now we introduce the following definition:

Definition: A non analytic complex function f on a region D is said to be fundamental complex function on D iff the only singularities of f in D are essential singular points.

Notation: We denote the set of all fundamental complex functions on a region D by $FCF(D)$.

Theorem (1): Let f be a complex valued function . If $E(f)=\phi$ on C^{∞} , then f has no removable singular points in the extended complex plane C^{∞}.

Proof: Since $E(f)=\phi$, then f has no essential singular points in C^{∞}, it follows that $f \in M(C^{\infty})$ (By Theorem 3.1 in [2]) , therefore f has no removable singular points in C^{∞}.

Theorem (2): Let f be an analytic function on a region D except for some singular points in D.

If $f \in FCF(D)$, then there is no curve γ in D such that $f=0$ on γ.

Proof: Assume that there exist a curve γ in D such that $f=0$ on γ, it follows from [7] that $f=0$ on $D-E(f)$, it follows that all singular points in D are removable.

Which is contradiction that $f \in FCF(D)$.

Thus there is no curve γ in D such that $f=0$ on γ.

Theorem (3): Let $f \in A(D-B)$, where B is the set of all singular points of f in a region D.

If $z \in d(N(f))-d(B)$ for all $z \in B$, then $f \in FCF(D)$.

Proof: $z \in B$ implies that $z \notin d(B)$.

So that there exists $r>0$ such that $(N(z,r)-\{z\}) \cap B=\varphi$, it follows that f has isolated singular points at z.

Since $f \in A(N(z,r)-\{z\})$, then any $z \in B$ is an essential singular point for f [8], it follows that $f \in FCF(D)$.

Corollary: Let $f \in A(D-B)$, where B is the set of all singular points of f in D .If $N(f)=\phi$ and $z \in d(P(f))-d(B)$ for all $z \in B$, then

$$\frac{1}{f} \in FCF(D).$$

Proof: Suppose that $z \in d(P(f))-d(B)$ for all $z \in B$, it follows from [7] that $z \in d(N(\frac{1}{f}))-d(B)$ for all $z \in B$.

Thus $\frac{1}{f} \in FCF(D)$ by Theorem (3).

Remark: The following theorems discovers the relation between harmonic functions and fundamental complex functions on the same region.

Theorem (4): Let w be an interior point of a region D , and let $U(x,y)$ be a real valued function satisfies the following conditions:

1. $U \in H(D-\{w\})$.
2. $(z-w)^m U \notin B(N(w,r)-\{w\})$ for all positive real numbers r and natural numbers m .
3. The conjugate period of U with respect to the curve $\partial N(w,r)$ is zero.

Then there exists $f \in FCF(D)$ such that $U=\mathrm{Re}(f)$ on $D-\{w\}$.

Proof: Since the conjugate period of U with respect to the curve $\partial N(w,r)$ is zero, then there is a harmonic conjugate $V(X,y)$ for $U(x,y)$ on $D-\{w\}$ [9], that is , $f=U+iV\in A(D-\{w\})$.

Since $(z-w)^m U\notin B(N(w,r)-\{w\})$ for all positive real numbers r and natural numbers m, it follows that neither w a pole nor removable singular point for f , it follows that $f\in FCF(D)$.

In the above theorem we have a single singular point of f . It can be generalized to a finite number of singular points as in the following Corollary:

Corollary: Let $U(x,y)$ be a real valued function on a region $D-B$, where B is anon-empty finite subset of $Int(D)$. Suppose that U satisfies the following conditions:

1. $U\in H(D-B)$.
2. For any $w\in B$ There exist $r_w>o$ $(z-w)^m U\notin B(N(w,r)-\{w\})$ for all $r<r_w$, and natural numbers m.
3. The conjugate periods of U with respect to all curves $\partial N(w,r)$; $w\in B$ are zero.

Then there exists $f\in FCF(D)$ such that $U=\mathrm{Re}(f)$ on $D-B$.

Proof: It can be proved by using Theorem (4).

Remark: The following Theorem explain that the reciprocal of the function of exponential type σ is not fundamental complex function.

This kind of function was defined in [10] as follows:

An entire function $f:C\to C$ is of exponential type σ ,that is, for every $\varepsilon>0$ there is a constant $k=k(\varepsilon,f)>0$ such that:

$$|f(z)|\le ke^{(\varepsilon+\sigma)|z|}$$

Theorem(5): If f is a function of exponential type σ , then $\frac{1}{f}\notin FCF(C)$.

Proof: f is a function of exponential type σ , then , it follows that for every $\varepsilon>0$ there is a constant $k=k(\varepsilon,f)>0$ such that $|f(z)|\le ke^{(\varepsilon+\sigma)|z|}$, it follows that:

$$\left|\frac{1}{f(z)}\right|\ge\frac{1}{k}e^{-(\varepsilon+\sigma)|z|} \qquad (1)$$

Suppose that $\frac{1}{f}\in FCF(C)$.

$\frac{1}{f}\in FCF(C)$ implies that there exists $w\in E(\frac{1}{f})$, it follows from [7] that there is a sequence $\{z_n\}$ such that $\{z_n\}$ converges to the number w and $\{\frac{1}{f(z_n)}\}$ converges to the number $\frac{1}{(k+1)^2}e^{-(\varepsilon+\sigma)|w|}$.

Which contradicts (1).

Therefore $\frac{1}{f}\notin FCF(C)$

Theorem(6):

Let $f\in A(D-\{w\})-B(D-\{w\})$, where $w\in Int((D)\bigcap d(N(f))$.

If f is not conformal mapping on $N(w,r)-\{w\}$ for all $r>0$, then $f\in FCF(D)$.

Proof: Since f is not conformal mapping on $N(w,r)-\{w\}$ for all $r>0$, then f has no pole at w (See Theorem 3.8 in [2]).

Since $f\notin B(D)$, then f has no removable singular point at w , and then $f\in FCF(D)$.

Theorem (7): If $f\in FCF(D)$ and $f=g\circ h$ such that g is a rational function, then $h\notin M(D)$.

Proof: Suppose that $h\in M(D)$.

Since g is a rational function, then $f \in M(D)$ [11].

This is contradiction that $f \in FCF(D)$.

Hence $h \notin M(D)$.

Theorem (8): If $h \in M(C^{\infty})$, $f = g \circ h$ and $E(g) = \phi$ on C^{∞}, then $f \in M(C^{\infty})$, where C^{∞} is the extended complex plane.

Proof: Since $E(g) = \phi$ on C^{∞}, then g has no essential singular points in C^{∞}, it follows [7] that g is a rational function.

Thus $f \in M(C^{\infty})$ [12].

Theorem (9): If $f \in M(D) - A(D)$, then $e^{f} \in FCF(D)$.

Proof: Suppose that $f \in M(D) - A(D)$, it follows that f has only poles in D.

Since $e^{z} \neq 0$ for all $z \in C$, then e^{-f} has no zeros in D, it follows from [1] that $e^{f} = \frac{1}{e^{-f}}$ has no poles in D.

It is easy to show that e^{f} has no removable singular points in D.

Hence $e^{f} \in FCF(D)$.

Theorem(10): If $f \in FCF(D)$, then $P(g \circ f) = \phi$ for any function g.

Proof: Suppose that $w \in E(f)$ on a region D. There is a sequence $\{z_n\}$ in D such that $\{z_n\}$ converges to w and $\{f(z_n)\}$ converges to c for any $c \in C \setminus \{w\}$. Since g is continuous in $N(w,r) - \{w\}$, for some r, then $\{g(f(z_n))\}$ converges to $g(c)$, it follows that $g \circ f$ has no pole at *w*, and then $P(g \circ f) = \phi$.

Theorem (11): If $w \in E(g)$ on D, then $g \circ f$ has no removable singular point at *w* in a region D.

Proof: Suppose that $g \circ f$ has a removable singular point at w, then:

$$\lim_{z \to w} (g \circ f)(z) = c \, ; c \in C \qquad (1)$$

Since $w \in E(g)$, then for any $k \neq c$ there is a sequence $\{z_n\}$ in D such that $\{z_n\}$ converges to w and $\{g(z_n)\}$ converges to k (See[7]).

This is contradiction with (1).

$\therefore$ $g \circ f$ has no removable singular point at *w*.

From now we study some groups of essential singular points on a region D with composition operation.

In [4] the set $\Gamma(X, a)$ was defined as the set of all conformal mapping $f : (X, a) \to (X, a)$ such that $f(a) = a$, especially for $X = C$ and $a = \infty$, the set $\Gamma(C, \infty)$ consists of elements of the form:

$$f(z) = a_1 z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z} + \ldots\ldots \quad \text{with}$$

$a_1 \neq 0$.

If we substitute $z - w$ instead of z, we get:

$$f(z) = a_1(z-w) + a_0 + \frac{a_{-1}}{z-w} + \frac{a_{-2}}{(z-w)^2} + \ldots. \quad (1)$$

If we denote the set of all functions of the form (1) by $\Gamma_w(C, \infty)$, then it forms a group with composition operation, which is a subset of $FCF(N(w,r))$ (See [11]).

Theorem(12): There exist a proper subset G of $FCF(D)$ such that $(G \cup \{I_D\}, \circ)$ is a non trivial subgroup of $(\Gamma_w(C, \infty), \circ)$.

Proof: Let G be the set of functions in $\Gamma_w(C, \infty)$ of the form:

$$f(z) = a_1 z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z} + \ldots\ldots \quad \text{such}$$

that the sequence $\{a_{-n}\}$ has infinite numbers of non-zero terms.

Since all members of G has infinite numbers of negative powers, then $G \subseteq FCF(D)$.

It is clear that G is a non trivial proper subset of $FCF(D)$.

Suppose that $f, g \in G \cup \{I_D\}$.

If either $f = I_D$ or $g = I_D$, then $f \circ g \in G \bigcup \{I_D\}$.

Suppose that both $f \neq I_D$ and $g \neq I_D$. From Theorem(10) and Theorem(11) we obtain that the function $f \circ g$ has neither pole nor removable singular point at w , therefore G is closed under composition operation. Suppose that $f \in G$, it follows that $f \in \Gamma_w(C,\infty)$, which implies that there exists $g \in \Gamma_w(C,\infty)$ such that $g \circ f = f \circ g = I_D$.

If $g \notin G$, then there are two cases for g :

1. g has a pole at w:

Since $w \in E(f)$, then there is a sequence $\{z_n\}$ of complex numbers such that $\{z_n\}$ converges to w, and $\{f(z_n)\}$ converges to w.

Since g has a pole at w , then $\{g(f(z_n))\}$ is divergent , it follows that $\{z_n\} = \{(g \circ f)(z_n)\}$ is divergent.

Which contradicts that $\{z_n\}$ converges to w.

2. g has a removable singular point at w:

Since $w \in E(f)$, then for every $c \in C \setminus \{w\}$, there is a sequence $\{z_n\}$ of complex numbers such that $\{z_n\}$ converges to w, and $\{f(z_n)\}$ converges to c , it follows that $\{z_n\} = \{(g \circ f)(z_n)\}$ converges to $g(c)$.

Since the converges point is unique then $g(c) = w$ for all $c \in C \setminus \{w\}$.

Thus the factor $a_1 = 0$ for g.

Which contradicts that $g \in \Gamma_w(C,\infty)$.

In either case we see that g has neither pole nor removable singular point at w.

Hence g has an essential singular point at w, it follows that $\{a_{-n}\}$ has infinite numbers of non-zero terms , it follows that $g \in G$.

$\therefore (G \bigcup \{I_D\},\circ)$ is a subgroup of $(\Gamma_w(C,\infty),\circ)$.

Theorem (13): Let D be a region.

There exist a subset G of $FCF(D) \bigcup \{I_D\}$ such that each of the following are true:

(1.) $(G,.)$ is a group.

(2.) $(G,.) \approx (C,+)$.

, where + and . are usual addition and usual multiplication respectively.

Proof:

Suppose that $G = \{g : g(z) = e^{\frac{c}{z-w}}, c \in C\}$.

It is clear from Theorem (9) that $G \subset FCF(D) \bigcup \{I_D\}$.

It is easy to show that $(G,.)$ forms a group.

Now we define a function $S : G \to C$ by:

$S(g) = \operatorname{Res}(g)$.

Suppose that $f, g \in G$.

The statement $f = g$ implies that $\operatorname{Res}(f,w) = \operatorname{Res}(g,w)$, which implies that $S(f) = S(g)$.

Thus S is Well-defined.

It is easy to show that S is onto.

Suppose that $f, g \in G$ such that $\operatorname{Res}(f,w) = c$ and $\operatorname{Res}(g,w) = k$.

$S(f) = S(g)$ implies that $\operatorname{Res}(f,w) = \operatorname{Res}(g,w)$.

Then we get $c = k$.

So that $\frac{c}{z-w} = \frac{k}{z-w}$,which yields $e^{\frac{c}{z-w}} = e^{\frac{k}{z-w}}$.

Hence $f = g$.

Therefore S is one to one.

Suppose that $f, g \in G$, then there exist $c, k \in C$ such that $\operatorname{Res}(f,w) = c$ and $\operatorname{Res}(g,w) = k$.

$$S(f.g) = \operatorname{Res}(e^{\frac{c}{z-w}}.e^{\frac{k}{z-w}}, w)$$

$$= \operatorname{Res}(e^{\frac{c+k}{z-w}}, w)$$

$= c + k$
$= \operatorname{Re} s(f,w) + \operatorname{Re} s(g,w)$
$= S(f) + S(g)$

Thus S is a homomorphism.

Hence $(G,.) \approx (C,+)$ by the definition of homomorphism in [9].

Corollary: Let D be a region. There is an infinite number of subsets G of $FCF(D) \bigcup \{I_D\}$ such that $(G,.)$ forms a group.

Proof: Since $(C,+)$ possesses an infinite number of subgroups , then $(G,.)$ also possesses an infinite number of subgroups by Theorem(14).

Finally There is an infinite number of subsets G of $FCF(D) \bigcup \{I_D\}$ such that $(G,.)$ forms a group.

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هەندێك سیفاتی نەخشە ئاڵۆزە بنجینەییەكان

فریاد حوسێن عەبدولقادر
بەشی بیركاری ، كۆلیژی پەروەردە زانستەكان ، زانكۆی سەلاحە ددین لە هەولێر ، هەرێمی كوردستانی عێراق

پوختە

ئامانجی ئەم توێژینەوەیە لێكۆڵینەوەیەكە لە جۆرێكی نوێ ی نەخشە كانە كە خالی ناوازە(تاك) دەگرێتە خۆ لەسەر ناوچەیەكی دیاریكراو لە روتەختی ئاڵۆز بۆ بە دەست هێنانی جەند سیفەتێكی ئەم جۆرە نەخشانە وە هەروەها لێكۆڵینەوەیەكی(جەبری)مان كردوە لەسە كۆمەڵەی ئەم جۆرە نەخشانە بۆ بە دەستهێنانی هەندێك لە بیردۆز و ئە نجامەكان.

بعض خواص الدوال المعقدة الاساسية

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الخلاصة

الهدف في هذا البحث هو دراسة نوع جديد من الدوال التي تمتلك نقطة مفردة أساسية على منطقة معينة في مستوي معقد للحصول على بعض خواص هذا النوع من الدوال ، وكذلك درسنا مجموعة هذا النوع من الدوال جبريا للحصول على بعض النظريات والنتائج.

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